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The role of an $L_2(\Omega)$-energy estimate in the theories of uniform stabilization and exact controllability for Schrödinger equations with Neumann boundary control *

R. Triggiani

ABSTRACT: The present paper deals with (linear) Schrödinger equations, of very general form, which are defined on a bounded domain $\Omega \subset \mathbb{R}^n$. With focus on these dynamics, we shall then discuss and analyze the specific and foundational topic of *a-priori* energy identities, with the goal of deriving control-theoretic implications. These will include the issue of optimal regularity, as well as the problems of exact controllability (by open loop controls) and of uniform stabilization (by closed loop feedback controls).

Key Words: Schrödinger equation, Neumann boundary control, exact controllability, uniform stabilization.

Contents

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0. Energy level estimates and their control-theoretic implications

The present paper deals with (linear) Schrödinger equations, of very general form, which are defined on a bounded domain $\Omega \subset \mathbb{R}^n$. With focus on these dynamics, we shall then discuss and analyze the specific and foundational topic of *a-priori* energy identities, with the goal of deriving control-theoretic implications. These will include the issue of optimal regularity, as well as the problems of exact controllability (by open loop controls) and of uniform stabilization (by closed loop feedback controls). In all cases, results are not obtained directly, but rather by duality following the strategy introduced in [L-L-T.1] in the case of optimal regularity for second-order hyperbolic equations with Dirichlet boundary control and, subsequently, by corresponding control-theoretic results for wave equation models [Tr.1], [Tr.2], [L-T.1], [L-T.2], [H.1], [Lio.1]. See an account in [G-L-L-T.1].

Returning to Schrödinger equations, we shall note that, at first, energy identities/estimates are obtained for sufficiently smooth solutions of the equations with no boundary conditions (B.C.) imposed: in this major step, they are the identities/estimates themselves that contain explicit boundary traces terms. Only in a second phase, B.C. are imposed on the solutions: either homogeneous B.C. or else B.C. of dissipative character. This way, one specializes the original identities/estimates to inequalities which—by duality—imply (i) either regularity results of corresponding mixed problems ("direct inequalities"), or else exact controllability or uniform stabilization with (open loop or, respectively, closed loop) boundary controls ("inverse inequalities"). The topological level of the regularity/control-theoretic results depends critically on the topological level of the *a-priori* identity/estimates; and these, in turn, depend critically on the technical tools employed to achieve them.

$H^1(\Omega)$-level estimates. It has been known for well over a decade [L-T.3], [M.1] that the 'natural' energy level of Schrödinger equations is $H^1(\Omega)$: This means that 'natural and effective' energy methods produce an energy identity at the $H^1(\Omega)$-topological level.

We reinforce once more the point made before, that the actual achievements of topological energy identities for Schrödinger equations, as well as the subsequent analysis thereof, were inspired by, and followed naturally, the prior development of second-order hyperbolic equations in [L-L-T.1], where the natural energy level was $H^1(\Omega) \times L_2(\Omega)$. It is useful to group the relevant identities/inequalities into two categories: (i) *pointwise* Carleman-type *inequalities* and (ii) *integral Carleman-type inequalities*.

(i) Pointwise inequalities (expressed originally pointwise, for each time instant t and each value x of the space coordinate) lead, after integration, to integral-type inequalities. However, they come with an additional, critical advantage in that pointwise (Carleman-type) inequalities contain no lower-order term ($\ell.o.t.$). This feature has a very helpful beneficial implication. This is that the same train of arguments provides, in one shot, both control-theoretic estimates of exact controllability/uniform stabilization, as well as new global uniqueness results for appropriate over-determined problems. This advantage is no small feat, given the low regularity of the variable coefficients of the equations, which will make it difficult to invoke results from the literature.

(ii) Integral-type inequalities are the results of using suitable 'multipliers.' These have been much extended from the classical multipliers of the mid- to late-eighties: $h\cdot(\nabla w)$ for canonical wave equations [L-L-T.1], $h\cdot(\nabla\overline{w})$ for canonical Schrödinger equations [L-T.3], [M.1], etc., where $h(x)$ is a suitable vector field, and w is the wave solution, respectively the Schrödinger solution. See marked generalizations in [G-L-L-T.1]. Unfortunately (save special cases where $h(x)$ is a radial vector field $h(x) = (x - x_0)$), multipliers techniques do yield the required control-theoretic estimates polluted, however, by lower-order terms. To remove these, one needs to absorb them into the terms of the sought-after final estimate. This can be done, for instance, by use of a compactness-uniqueness argument (first introduced in [Lit.1] in the present control theory context). Compactness is typically no problem; however, uniqueness is a serious issue beyond the case of analyticity of the coefficients (Holmgren, Tataru [Ta.2], Hormander [Ho.1], ... Indeed, the global uniqueness results via pointwise Carleman estimates as discussed before in (i) are very handy here. A disadvantage of the strategy of absorbing $\ell.o.t.$ by a compactness–uniqueness proof is that the argument is generally indirect; that is, by contradiction, so that—in this step—one loses control of the constants involved. See also [K.1].

Henceforth, we shall focus on the $L_2(\Omega)$-level energy inequalities for Schrödinger equations, the topic of the present paper.

1. An a-priori energy estimate for a general Schrödinger equation at the $L_2(\Omega)$-level

Here below we shall consider smooth solutions of a general Schrödinger equation with a forcing term. Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with a sufficiently smooth boundary $\partial\Omega = \Gamma$, say of class C^2. We shall focus on the case dim $\Omega = n \geq 2$. We write $\Gamma = \overline{\Gamma_0 \cup \Gamma_1}$, where Γ_0 will be the uncontrolled or unobserved part of Γ, and Γ_1 is the controlled or observed part of Γ, and Γ_1 is the controlled or observed part of Γ, both relatively open in Γ. We let ν be the outward unit normal along Γ. In Ω, we consider the following Schrödinger equation [Tr.3], [T-Y.1]:

$$iz_t + \mathcal{A}z = F(z) + f; \tag{1.1}$$

$$\mathcal{A}z = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\ \frac{\partial z}{\partial x_j}\right);\ \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq a\sum_{i=1}^{n}\xi_i^2; \tag{1.2}$$

$$F(x) = ir(t,x)\cdot\nabla z + q_0(t,x)z, \tag{1.3}$$

satisfying, in addition:

$$\text{either } z|_{\Sigma_0} \equiv 0, \quad \text{in which case } \nabla d\cdot\nu \leq 0 \text{ on } \Gamma_0; \tag{1.4a}$$

$$\text{or else } \left.\frac{\partial z}{\partial\nu}\right|_{\Sigma_0} \equiv 0, \quad \text{in which case } \nabla d\cdot\nu \equiv 0 \text{ on } \Gamma_0, \tag{1.4b}$$

where $d(x)$ is a non-negative, real-valued, strictly convex function: $\overline{\Omega} \to R^+$. Thus, the matrix

$$\mathcal{H}_d(x) = \left[\frac{\partial^2 d}{\partial x_i \partial x_j}\right]_{i,j=1}^n \text{ satisfies } \mathcal{H}_d x \cdot \bar{x} \geq \rho |x|^2, \; \rho > 0, \; x \in \mathbb{R}^n, \tag{1.5}$$

for some $\rho > 0$. The simplest example is $d(x) = \frac{1}{2}\|x - x_0\|^2$, the square of the distance from a suitable fixed point $x_0 \in \mathbb{R}^n$, whereby then $\nabla d = (x - x_0)$ and $\mathcal{H}_d = I$, the $n \times n$ identity matrix, with $\rho = 1$. Regarding the coefficients $r(t,x) \in \mathbb{R}^n$ and $q_0(t,x)$ (scalar) in Eqn. (1.1), we assume the following hypotheses:

(A.1) q_0 is a complex-valued function on $[0,T] \times \Omega$, while $r(t,x)$ is a *real-valued* vector field on $\mathbb{R}_t \times \Omega$ (structural property [R-S.1] of "magnetic potential") satisfying the following regularity hypotheses:

$$q_0 \in L_\infty(Q), \; |\nabla q_0| \in L_\infty(Q), \; r \in L_\infty(0,T;\mathbb{R}^n), \tag{1.6}$$

so that for the energy level term F, we have

$$|F(z)|^2 \leq C_T\{|\nabla z|^2 + |z|^2\}, \quad \forall \; (t,x) \in Q \text{ a.e.} \tag{1.7}$$

The $L_2(\Omega)$ energy level estimate is

Theorem 1.1. *Assume hypothesis (A.1). Let z be a solution of Eqn. (1.1) satisfying, in addition, either the Dirichlet case (1.4a), or else the Neumann case (1.4b). Let $T > 0$ be arbitrary. Finally, let $f \in L_2(0,T;L_2(\Omega))$. Then, the following inequality holds true: There exists a constant $C_T > 0$ such that*

$$\begin{aligned} &\int_0^T \left[\|z\|^2_{L_2(\Omega)} + \|z_t\|^2_{H^{-2}(\Omega)}\right] dt + \|z(0)\|^2_{L_2(\Omega)} + \|z_t(0)\|^2_{H^{-2}(\Omega)} \\ &\quad \leq \; C_T \Bigg\{ \|z\|^2_{L_2(\Sigma_1)} + \left\| \left.\frac{\partial z}{\partial \nu}\right|_{\Gamma_1} \right\|^2_{H_a^{-1}(\Sigma_1)} \\ &\qquad + \int_0^T \int_{\Gamma_1} \left| \left.\frac{\partial z}{\partial \nu}\right|_{\Gamma_1} \right| |z| d\Gamma_1 dt + \|z\|^2_{H^{-1}(Q)} + \|f\|^2_{L_2(Q)} \Bigg\}, \end{aligned} \tag{1.8}$$

where $H_a^{-1}(\Sigma_1)$ is the dual space to the anisotropic space $H_a^1(\Sigma_1)$, with respect to the pivot space $L_2(\Sigma_1)$:

$$H_a^{-1}(\Sigma_1) = (H_a^1(\Sigma_1))'; \; H_a^1(\Sigma_1) \equiv H^{\frac{1}{2}}(0,T;L_2(\Gamma_1)) \cap L_2(0,T,H^1(\Gamma_1)). \qquad \Box \tag{1.9}$$

Remark 1.1. The natural energy level for the Schrödinger equation is the $H^1(\Omega)$-level, *not* the $L_2(\Omega)$-level. Indeed, the proof of the energy estimate (1.8) at the $L_2(\Omega)$-level for (1.1) requires a heavy use of pseudo-differential/micro-local analysis machinery [L-T-Z.2, Sect. 10], to shift the more natural $H^1(\Omega)$-level energy estimate to the $L_2(\Omega)$-level. The proof in [L-T-Z.2, Sect. 10] refers specifically to an Euclidean domain Ω in $\mathbb{R}^n$, $\partial\Omega = \overline{\Gamma_0 \cup \Gamma_1}$. It is based on partition of unity of Ω, flattening the boundary locally, and consequent analysis in the half-space, by taking, as a starting point, the *a-priori* energy estimate at the $H^1(\Omega)$-level from [Tr.3], in the Euclidean case. However, it was already noted in [L-T-Z.2, Remark 2.6.2], that by taking this time, as a starting point, the *a-priori* $H^1(\Omega)$-energy level estimate in the Riemannian case from [T-Y.1], the same proof works also in the case where Ω is an open, bounded, connected set Ω of an n-dimensional, Riemannian manifold M, as in Remark 2.1 below. □

Here below, we shall review two critical recent consequences of estimate (1.8): an exact controllability result in $L_2(\Omega)$ with Neumann $L_2(\Sigma_1)$-boundary open-loop control (Section 2); and uniform stabilization results in $L_2(\Omega)$ with linear and non-linear dissipative boundary terms (feedback controls) in the Neumann B.C.(Section 3). Section 2 is a review of paper [Tr.3]; Section 3 is a review of paper [L-T.9]. We shall confine here to statements of results as well as to illustrative examples, while referring to [Tr.3] and [L-T.9] for the full analysis.

2. Consequence #1: exact controllability in the state space $L_2(\Omega)$ with $L_2(0,T;L_2(\Gamma_1))$-Neumann control [Tr.3]

2.1. MODEL AND MAIN RESULT. Let the triple $\{\Omega, \Gamma_0, \Gamma_1\}$ be as in Section 1. In this section, we consider the following mixed Schrödinger problem in the (complex-valued) unknown $w(t,x)$ defined on Q,

$$\begin{cases} iw_t + \Delta w = F(w) & \text{in } Q \equiv (0,T] \times \Omega; & \text{(2.1a)} \\ w(0,\cdot) = w_0 & \text{in } \Omega; & \text{(2.1b)} \\ \text{either } w|_{\Sigma_0} \equiv 0, \text{ or else } \left.\dfrac{\partial w}{\partial \nu}\right|_{\Sigma_0} \equiv 0, & \text{in } \Sigma_0 = (0,T] \times \Gamma_0; & \text{(2.1c)} \\ \left.\dfrac{\partial w}{\partial \nu}\right|_{\Sigma_1} \equiv u, & \text{in } \Sigma_1 \equiv (0,T] \times \Gamma_1, & \text{(2.1d)} \end{cases}$$

with Neumann boundary control $L_2(0,T;L_2(\Gamma_1))$. In the case $w|_{\Sigma_0} \equiv 0$, we also assume $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$. In (1.1a), we have set

$$F(w) = -ir(t,x) \cdot \nabla w + q_0(t,x)w, \tag{2.2}$$

as in (1.3), with $r(t,x)$, $q_0(t,x)$ subject to assumption (A.1).

The following is an exact controllability result in the state space $L_2(\Omega)$ within the class of $L_2(0,T;L_2(\Gamma_1))$-Neumann controls, where $T > 0$ is preassigned arbitrarily small.

Theorem 2.1. *With reference to the mixed problem (2.1a-b-c-d), assume that the strictly convex function $d(x)$ in (1.5) satisfies: $\nabla d \cdot \nu \leq 0$ on Γ_0 in the case of the Dirichlet B.C. $w|_{\Sigma_0} \equiv 0$ in (2.1c); and $\nabla d \cdot \nu \equiv 0$ on Γ_0 in the case of the Neumann B.C. $\frac{\partial w}{\partial \nu}|_{\Sigma_0} \equiv 0$ in (2.1c). Let the coefficients of F satisfy assumptions (A.1) =*

(1.6). Then, the w-problem (2.1a-b-c-d) is exactly controllable in the following sense. Let $T > 0$ be arbitrary. Given $w_0 \in L_2(\Omega)$ [respectively, $w_1 \in L_2(\Omega)$], there exists a boundary control $u \in L_2(0, T; L_2(\Gamma_1))$ such that the corresponding solution of the w-problem (2.1) due to the data $\{w_0, u\}$ [respectively, due to the data $\{w_0 = 0, u\}$] satisfies $w(T) = 0$ [respectively, $w(T) = w_1$].

Remark 2.1. Theorem 2.1 holds true also in the following Riemannian setting [Tr.4]. Let M be a complete n-dimensional, Riemannian manifold of class C^3 with C^3-metric $g(\,\cdot\,,\,\cdot\,) = \langle\,\cdot\,,\,\cdot\,\rangle$ and squared norm $|X|^2 = g(X, X)$. We shall denote it by (M, g). Let Ω be an *open, bounded, connected* set of M with smooth boundary (say, of class C^2) $\partial\Omega \equiv \Gamma = \overline{\Gamma_0 \cup \Gamma_1}$. Here, Γ_0 is the uncontrolled or unobserved part of Γ and Γ_1 is the controlled or observed part of Γ, both relatively open in Γ. We let ν denote the outward unit normal field along the boundary Γ. Further, we denote by ∇ the gradient, by D the Levi-Civita connection, by D^2 the Hessian, by $\Delta = \operatorname{div}(\nabla)$ the Laplace (Laplace-Beltrami) operator [Le.1, p. 55, p. 83, p. 141], [Do.1, p. 28, pp. 43–44, p. 54, p. 68].

The normal derivative conditions $\frac{\partial w}{\partial \nu}|_{\Sigma_0} \equiv 0$ and $\frac{\partial w}{\partial \nu}|_{\Sigma_1} \equiv u$ read now

$$\langle Dw, \nu\rangle|_{\Sigma_0} \equiv 0 \quad \text{and} \quad \langle Dw, \nu\rangle|_{\Sigma_1} \equiv u, \tag{2.3}$$

respectively. The function $F(w)$ in (2.2) is now written as

$$F(w) = -i\langle R(t,x), Dw\rangle + q_0(t,x)w, \tag{2.4}$$

where the coefficients are subject to the following assumptions:

$$q_0 \in L_\infty(Q), \quad R \in L_\infty(0, T, \mathcal{X}(M)) \qquad \text{[He.1]}, \tag{2.5a}$$

so that for the energy level term F, we have

$$|F(w)|^2 \le C_T\{|Dw|^2 + |w|^2\}, \qquad \forall\ (t,x) \in Q \text{ a.e.}, \tag{2.5b}$$

where $Dw = \nabla w$ for the scalar function w. Thus, $Dw \in \mathcal{X}(M) =$ the set of all C^2 complex-valued vector fields on M. Next, recall that the covariant differential (a 2-0 tensor T_2^0) of $R \in \mathcal{X}(M)$ determines a bilinear form on $TM \times TM$, for each $x \in M$, defined by $DR(X, Y) = \langle D_X R, Y\rangle_g$. Then, we require that:

$$\begin{cases} |DR(X,Y)| = |\langle D_X R, Y\rangle| \le C|X|\,|Y|, \quad 0 \le t \le T, \\ \text{or } DR \in L_\infty(0, T; T_2^0); \end{cases} \tag{2.5c}$$

and moreover,

$$|Dq_0| \in L_\infty(Q). \tag{2.5d}$$

$\square$

2.2. THE ADJOINT PROBLEM AND THE EQUIVALENT COI UNDER THE WORKING ASSUMPTION $\langle r \cdot \nu \rangle \equiv 0$ ON Γ_1 (RESP. ON Γ). As in the case for most of the exact controllability results for hyperbolic and Petrowski-type evolution equations in the literature, the proof of Theorem 2.1 is by duality: that is, it consists of establishing the equivalent continuous observability inequality [L-T.3], [Tr.4], [T-Y.1], [G-L-L-T.1]. (An exception is the direct work of W. Littman [Lit.1], [Lit-Ta.1]). In our present case, establishment of the continuous observability inequality in Section 3 relies critically on the $L_2(\Omega)$-energy level estimate (1.8).

The goal of the present subsection is two-fold. First, we shall seek to establish the PDE system which is obtained by duality or transposition over the mixed control problem (2.1a–d). This is the φ-problem (2.12) below. To this end, we shall make a temporary working assumption, (A.2) = (2.8) below, to be later removed in Section 2.4. Second, we shall obtain the relevant Continuous Observability Inequality (COI) for the φ-problem (2.12) which is *equivalent* to the exact controllability property of the w-problem in (2.1), as spelled out in the statement of Proposition 2.3. We begin by setting, for short

$$\mathcal{A}w \equiv i\Delta w - r(t,x) \cdot \nabla w - iq_0(t,x)w, \tag{2.6}$$

with $r(t,x)$ the *real-valued* vector field on $\mathbb{R}_t \times \Omega$, as in assumption (2.2), (A.1). With reference to problem (2.1), define the operator $A : L_2(\Omega \supset \mathcal{D}(A) \to L_2(\Omega)$ (depending on t), by

$$Aw \equiv \mathcal{A}w, \ \mathcal{D}(A) \equiv \left\{ w \in H^2(\Omega) : w|_{\Gamma_0} = \left.\frac{\partial w}{\partial \nu}\right|_{\Gamma_1} \equiv 0 \right\}, \tag{2.7a}$$

or else

$$\mathcal{D}(A) \equiv \left\{ w \in H^2(\Omega) : \left.\frac{\partial w}{\partial \nu}\right|_{\Gamma} \equiv 0 \right\}. \tag{2.7b}$$

Throughout this section, we shall impose the following *working assumption*:
(A.2)

$$\begin{cases} \text{either } r(t,x) \cdot \nu \equiv 0 \text{ on } \Gamma, & \text{if } \left|\frac{\partial w}{\partial \nu}\right|_{\Gamma_0} \equiv 0, \quad (2.8a) \\ \text{or else } r(t,x) \cdot \nu \equiv 0 \text{ on } \Gamma_1, & \text{if } |w|_{\Gamma_0} \equiv 0. \quad (2.8b) \end{cases}$$

This assumption will facilitate the analysis in establishing Theorem 2.1 at first. Later on, in Section 2.4, we shall dispense with assumption (A.2) = (2.8), by means of a natural change of variable, as in [L-T-Z.2, Appendix A, Proposition A.4, Eqn. (A.18), p. 107], whereby the geometrical assumption (A.2) = (2.8) will be satisfied by the new variable and exact controllability in the original variable will be equivalent to exact controllability in the new variable. Thus, for $w, \overline{\varphi} \in H^1(\Omega)$, under both assumptions (A.2) = (2.8a) and (A.2) = (2.8b), we have from the divergence (Green) formula:

$$\int_\Omega r \cdot \nabla w \overline{\varphi} \, d\Omega = -\int_\Omega w \ \mathrm{div}(\overline{\varphi} r) d\Omega. \tag{2.9}$$

The adjoint operator A^* of A under (A.2) = (2.8). The $L_2(\Omega)$-adjoint of the operator A in (2.7), subject to either (A.2) = (2.8a) or (A.2) = (2.8b), is:

$$A^*\varphi = -i\Delta\varphi + r(t,x)\cdot\nabla\varphi + i\tilde{q}_0\varphi,\ \tilde{q}_0 = \overline{q}_0 - \text{ div } r \in L_\infty(Q), \tag{2.10a}$$

and either

$$\mathcal{D}(A^*) = \mathcal{D}(A) = \left\{\varphi \in H^2(\Omega) : \varphi|_{\Gamma_0} = \left.\frac{\partial\varphi}{\partial\nu}\right|_{\Gamma_1} \equiv 0\right\}, \tag{2.10b}$$

in case (2.7a), or else

$$\mathcal{D}(A^*) = \mathcal{D}(A) = \left\{\varphi \in H^2(\Omega) : \left.\frac{\partial\varphi}{\partial\nu}\right|_{\Gamma} \equiv 0\right\}, \tag{2.10b$'$}$$

in case (2.7b). A direct computation using either assumption (A.2) = (2.8a), or else (A.2) = (2.8b), hence identity (2.9) in both cases, yields in fact, starting from (2.1):

$$(Aw,\varphi)_{L_2(\Omega)} = \int_\Omega (Aw)\overline{\varphi}\, d\Omega = \int_\Omega w\overline{(A^*\varphi)} d\Omega = (w, A^*\varphi)_{L_2(\Omega)},$$
$$w,\varphi \in \mathcal{D}(A) = \mathcal{D}(A^*). \tag{2.11}$$

The problem adjoint to (2.1a–d). On the basis of the operator A^* in (2.10) (under (A.2) = (2.8)), we consider the problem

$$\varphi_t = -A^*\varphi,\ \varphi(T) = \varphi_0; \begin{cases} \varphi_t = i\Delta\varphi - r\cdot\nabla\varphi - i\tilde{q}_0\varphi, & \text{in } Q; \quad (2.12a)\\ \varphi|_{t=T} = \varphi_0, & \text{in } \Omega; \quad (2.12b)\\ \text{either } \varphi|_{\Sigma_0} \equiv 0, \text{ or } \left.\frac{\partial\varphi}{\partial\nu}\right|_{\Sigma_0} \equiv 0, & \text{in } \Sigma_0; \quad (2.12c)\\ \left.\frac{\partial\varphi}{\partial\nu}\right|_{\Sigma_1} \equiv 0, & \text{in } \Sigma_1. \quad (2.12d)\end{cases}$$

When the I.C. $w_0 = 0$ in (2.1b), then the φ-problem (2.12a–d) is the adjoint to the control w-problem (2.1a–d). More precisely, we have:

Proposition 2.2. *With reference to problems (2.1) and (2.12), assume (A.1), (A.2). The closed map*

$$\mathcal{L}_T:\ \{w_0 = 0, u\} \to \mathcal{L}_T u = w(T),\ \textit{from } L_2(\Sigma_1) \supset \mathcal{D}(\mathcal{L}_T) \textit{ to } L_2(\Omega), \tag{2.13}$$

and the map

$$\mathcal{L}_T^*:\ \varphi_0 \to \mathcal{L}_T^*\varphi_0 = -i\varphi(\,\cdot\,;\varphi_0)|_{\Sigma_1} \textit{ from } L_2(\Omega) \supset \mathcal{D}(\mathcal{L}_T^*) \textit{ to } L_2(\Sigma_1), \tag{2.14}$$

are adjoint of each other: for $u \in \mathcal{D}(\mathcal{L}_T)$ and $\varphi_0 \in \mathcal{D}(\mathcal{L}_T^*)$,

$$(\mathcal{L}_T u, \varphi_0)_{L_2(\Omega)} = (w(T), \varphi_0)_{L_2(\Omega)} = \int_\Omega w(T)\overline{\varphi}_0 d\Omega$$

$$= \int_0^T \int_{\Gamma_1} u\overline{(-i\varphi)} d\Sigma_1 = (u, -i\varphi)_{L_2(\Sigma_1)} = (u, \mathcal{L}_T^* \varphi_0)_{L_2(\Sigma_1)}. \tag{2.15}$$

Proof. Multiply Eqn. (2.1a) by $\overline{\varphi}$ and integrate by parts over Q, invoking $w_0 = 0$ and the B.C. (2.1c–d) for w, and (2.12a–d) for φ. Details are straightforward using (2.9). □

Duality between exact controllability of the w-problem (2.1) with $w_0 = 0$ and continuous observability of the φ-problem (2.12). Exact controllability of problem (2.1a–d) with $w_0 = 0$, as spelled out in the statement of Theorem 2.1, over the interval $[0, T]$ on the state space $L_2(\Omega)$, within the class of Neumann-boundary controls $L_2(0, T; L_2(\Gamma_1))$ means precisely that the (closed) map $\mathcal{L}_T$ in (2.13) satisfies

$$\mathcal{L}_T : L_2(0, T; L_2(\Gamma_1)) \supset \mathcal{D}(\mathcal{L}_T) \stackrel{\text{onto}}{\longrightarrow} L_2(\Omega). \tag{2.16}$$

Equivalently then [Ta-La.1, p. 235], the adjoint operator $\mathcal{L}_T^*$ in (2.14) is bounded below: there exists a constant $c_T' > 0$ such that

$$\|\mathcal{L}_T^* z\|_{L_2(0,T;L_2(\Gamma_1))} \geq c_T' \|z\|_{L_2(\Omega)}, \quad z \in \mathcal{D}(\mathcal{L}_T^*). \tag{2.17}$$

Recalling (2.14) for $\mathcal{L}_T^*$, we obtain the Continuous Observability Inequality (COI) in terms of the adjoint φ-problem (2.12), under the working assumption (A.2).

Proposition 2.3. *Assume (A.1), (A.2). The exact controllability property of problem (2.1a–d) spelled out in the statement of Theorem 2.1 (in symbols: property (2.16)) is equivalent to the following COI: There exists a constant $c_T > 0$, independent of φ_0, such that the solution of problem (2.12) satisfies*

$$\|\varphi_0\|^2_{L_2(\Omega)} \leq c_T \int_0^T \int_{\Gamma_1} |\varphi(\,\cdot\,; \varphi_0)|^2 d\Sigma_1, \ \varphi_0 \in L_2(\Omega), \tag{2.18}$$

whenever the right-hand side of (2.18) is finite.

2.3. PROOF OF THE COI (2.18) UNDER (A.2). The goal of this section is to establish the *Continuous Observability Inequality* (2.18) for the adjoint φ-problem (2.12a-b-c-d), under the working assumption (A.2).

Regularity. First, however, we need to establish the regularity of problem (2.12).

Theorem 2.4. *Let $T > 0$ be arbitrary. Assume (A.1), (A.2) (so that $\tilde{q}_0 \in L_\infty(Q)$, see (2.10a)). With reference to the φ-problem (2.12a–d) with $\varphi_0 \in L_2(\Omega)$, we have that the solution map*

$$\varphi_0 \in L_2(\Omega) \to \varphi \in C([0, T]; L_2(\Omega)) \tag{2.19}$$

is continuous.

The proof is given in [Tr.4, Section 4].

Continuous Observability Inequality. We next establish inequality (2.18) at first under the working assumption (A.2). This will be removed in Section 2.4.

Theorem 2.5. *Let $T > 0$ be arbitrary. With reference to the φ-problem (2.12a–d) with $\varphi_0 \in L_2(\Omega)$, assume (A.1) for the coefficients r and $\tilde{q}_0$, so that $\tilde{q}_0 \in L_\infty(Q)$. Further, assume that the strictly convex $d(x)$ in (1.5) satisfies $\nabla d \cdot \nu \le 0$ on Γ_0 in case of the Dirichlet B.C. $\varphi|_{\Sigma_0} \equiv 0$ in (2.12c); and $\nabla d \cdot \nu \equiv 0$ on Γ_0 in case of the Neumann B.C. $\frac{\partial \varphi}{\partial \nu}|_{\Sigma_0} \equiv 0$ in (2.12c). Further, assume the working assumption (A.2)= (2.8). Then, the following estimate holds true: There exists a constant $c_T > 0$, independent of φ_0, such that*

$$\|\varphi_0\|^2_{L_2(\Omega)} \le c_T \int_0^T \int_{\Gamma_1} |\varphi|^2 d\Sigma_1, \tag{2.20}$$

whenever the right-hand side of (2.20) is finite.

Proof. Step 1. One first shows the estimate

$$\int_0^T \int_{\Gamma_1} |\varphi|^2 d\Sigma_1 + \|\varphi\|^2_{H^{-1}(Q)} \ge \tilde{c}_T \|\varphi_0\|^2_{L_2(\Omega)}, \tag{2.21}$$

for $\tilde{c}_T > 0$ independent of φ_0, which is inequality (2.20) polluted by an interior lower-order term. The key inequality (2.21) is readily seen to be a direct application of estimate (1.8) (with $f \equiv 0$) of Theorem 1.1, after using the homogeneous Neumann B.C. in (2.12c).

Step 2. Naturally, for $c_T > 0$ independent of φ_0, (2.21) implies *a-fortiori*

$$\int_0^T \int_{\Gamma_1} |\varphi|^2 d\Sigma_1 + \|\varphi\|^2_{L_\infty(0,T;H^{-1}(\Omega)} \ge c_T \|\varphi_0\|^2_{L_2(\Omega)}, \tag{2.22}$$

as the interior term in (2.22) dominates the interior term in (2.21).

Step 3. Next, we need to absorb the interior *l.o.t.* $\varphi \in L_\infty(0, T; H^{-1}(Q))$ by a compactness/uniqueness argument, as usual. The uniqueness part is the delicate point. Thus, one needs to establish the following result in order to complete the proof of Theorem 2.5.

Lemma 2.6. *Assume the hypotheses of Theorem 2.5, and let φ be a solution of problem (2.12) satisfying inequality (2.22). Then, in fact,*

$$\|\varphi\|^2_{L_\infty(0,T;H^{-1}(\Omega))} \le k_T \int_0^T \int_{\Gamma_1} |\varphi|^2 d\Sigma_1, \tag{2.23}$$

for a constant $k_T > 0$ independent of φ_0.

The proof is given in [Tr.4, Section 3] in the more general Riemannian setting.

2.4. REMOVAL OF ASSUMPTION (A.2) = (2.8). In this section we complete the proof of Theorem 2.1 in its full strength, by removing the working assumption (A.2) = (2.8). To this end, we perform in the original w-problem a change of variable as the one taken in [L-T-Z.2], [Tr.4, Eqn. (4.3)] for the dual φ-problem (2.12a–d); that is, we set

$$y(t,x) = e^{-\frac{i}{2}p(t,x)}w(t,x), \tag{2.24}$$

for a smooth real function $p(t,x)$. Then, the problem in y corresponding to the w-problem (1.1a–d) is

$$\begin{cases} y_t = i\Delta y - [r(t,x) + \nabla p(t,x)] \cdot \nabla y - iq_1(t,x)y & \text{in } Q; \quad (2.25a) \\ \text{either } y|_{\Sigma_0} \equiv 0, \text{ or else } \left[\dfrac{\partial y}{\partial \nu} + \left(\dfrac{i}{2}\dfrac{\partial p}{\partial \nu}\right) y\right]_{\Sigma_0} \equiv 0 & \text{in } \Sigma_0; \quad (2.25b) \\ \dfrac{\partial y}{\partial \nu} + \left(\dfrac{i}{2}\dfrac{\partial p}{\partial \nu}\right) y = e^{-\frac{i}{2}p(t,x)}u & \text{in } \Sigma_1, \quad (2.25c) \end{cases}$$

where $\tilde{r}(t,x) = r(t,x) + \nabla p(t,x)$ is a real-valued vector field on $\mathbb{R}_t \times \Omega$, satisfying the same assumptions $\tilde{r} \in L_\infty(0,T;\mathbb{R}^n)$ in (A.1) = (1.6). Similarly, q_1, which is given by [Tr.1, Eqn. (4.5)]

$$q_1 = \left[q_0 + \frac{1}{2}p_t - \frac{i}{2}\Delta p - \frac{1}{4}|\nabla p|^2 + \frac{1}{2}\langle r + \nabla p, \nabla p\rangle\right] \in L_\infty(Q), \tag{2.26}$$

satisfies $q_1 \in L_\infty(Q)$, $|\nabla q_1| \in L_\infty(Q)$, as required by (1.6).

Moreover, the original real-valued vector field $r(t,x)$ in (1.3), it is always possible to select, in infinite many ways, a smooth real function p such that

$$\tilde{r} \cdot \nu|_\Sigma = [r \cdot \nu + \nabla p \cdot \nu]_\Sigma = 0.$$

Thus, to the y-problem (2.25), we can apply the same duality argument used in subsection 2.3 with respect to the original w-problem in (2.1) (except for the noncritical fact that the B.C. (2.25) for y is of Robin-type, while the B.C. (2.1c) for w is of Neumann-type. Accordingly, by Section 2.3, Theorem 2.5 is applied to the dual of the y-problem (2.25). We conclude that the y-problem (2.25) is exactly controllable on the state space $L_2(\Omega)$ by means of $L_2(0,T;L_2(\Gamma_1))$ controllers of the type $e^{-\frac{i}{2}p(t,x)}u(t,x)$. But then by (2.24), the w-problem (2.1a-b-c) is likewise exactly controllable on the state space $L_2(\Omega)$, by means of controllers of the type u in $L_2(0,T;L_2(\Gamma_1)$. Theorem 2.1 is proved.

3. Consequence #2: sharp uniform decay rates at the $L_2(\Omega)$-level with nonlinear boundary dissipation in the Neumann B.C. [L-T.9]

The goal of the present section is to provide a uniform boundary stabilization result at the $L_2(\Omega)$-energy level (Theorems 3.1 and 3.3) for a multi-dimensional Schrödinger equation model in feedback form, with nonlinear dissipation in the Neumann-boundary conditions. The model is given in Eqns. (3.3a–c) in subsection

3.2 below. The aforementioned boundary feedback energy decay result with nonlinear boundary dissipation is, in turn, motivated by, and a generalization of, the corresponding linear problem, which was recently obtained in [L-T-Z.2, Sect. 11]. Accordingly, we need to review briefly such linear result first.

3.1. LINEAR CASE: CONSERVATIVE OPEN-LOOP AND DISSIPATIVE CLOSED-LOOP SCHRÖDINGER PROBLEMS IN $L_2(\Omega)$ [L-T-Z.2]. Recently the following problem was considered in [L-T-Z.2, Sect. 11]. Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be an open bounded domain with sufficiently smooth boundary $\partial\Omega = \Gamma$ of class C^2, $\Gamma = \overline{\Gamma_0 \cup \Gamma_1}$, $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$, $\Gamma_0 \neq \emptyset$. In Ω, we consider the following two Schrödinger problems

$$\begin{cases} iy_t + \Delta y = 0; \\ y(0,\,\cdot\,) = y_0; \\ y|_{\Gamma_0} \equiv 0, \left.\frac{\partial y}{\partial \nu}\right|_{\Gamma_1} \equiv u; \end{cases} \begin{cases} iv_t + \Delta v = 0 & \text{in } Q = (0,T] \times \Omega; \\ v(0,\,\cdot\,) = v_0 & \text{in } \Omega; \\ v|_{\Gamma_0} \equiv 0, \left.\frac{\partial v}{\partial \nu}\right|_{\Gamma_1} \equiv iv & \text{in } \Sigma_k = (0,T] \times \Gamma_k, k = 0, 1, \end{cases} \begin{matrix} (3.1a) \\ (3.1b) \\ (3.1c) \end{matrix}$$

$\nu(x)$ = outward unit (real) normal at $x \in \Gamma_1$, where the v-problem can be viewed as a closed-loop version of the y-Neumann problem, with boundary control u on Γ_1 in the feedback form $u = iv$. For $u \equiv 0$, the y-problem is conservative ('energy' preserving), with skew-adjoint generator $\mathbb{A} = -\mathbb{A}^*$. Instead, in contrast, the v-problem is dissipative, as quantitatively stated in Theorem 3.1 below. The solution y or v is complex-valued. A comparison with [L-T.3], [M.1] is given below at the end of Section 3.3.

Remark 3.1. In (3.1c), with cosmetic changes, we could also allow $\Gamma_0 = \emptyset$ and $\Gamma \equiv \Gamma_1$. More importantly, in (3.1c) we could also include the case where the homogeneous Dirichlet B.C. $y|_{\Gamma_0} \equiv 0$, $v|_{\Gamma_0} \equiv 0$ on Γ_0 are replaced by the corresponding homogeneous Neumann B.C. $\frac{\partial y}{\partial \nu}|_{\Gamma_0} \equiv 0$, $\frac{\partial v}{\partial \nu}|_{\Gamma_0} \equiv 0$, respectively, in which case the condition $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$ is dispensed with. However, in this latter 'purely' Neumann case, more stringent geometrical conditions are called for: there exists a smooth strictly convex function $d : \overline{\Omega} \to \mathbb{R}$ such that the vector field $\ell(x) = \nabla d(x)$ satisfies: $\ell \cdot \nu \equiv 0$ on Γ_0 (instead of the more relaxing condition $\ell \cdot \nu \leq 0$ on Γ_0 in the Dirichlet case, as assumed in Theorem 3.1 and (H.3) = (3.6) below). Various classes of $\{\Omega, \Gamma_0, \Gamma_1\}$ where such strictly convex function d may be constructed are given in [L-T-Z.1, Appx. A, pp. 287–307]. In particular, *a sufficient condition is that* Γ_0 *be convex or concave* [L-T-Z.1, Thm. A.4.1, p. 301] and [T-Y.2, Appx. B] (in a Riemannian setting). Same considerations apply to the nonlinear problem (3.3) below. □

Well-posedness and stabilization. See [L-T-Z.2, Sect. 11] and [L-T-Z.3, Sect. 11].

Theorem 3.1. (Well-posedness and strong stabilization in $L_2(\Omega)$ [L-T-Z.2, Thm. 11.1.1]) *With reference to the v-problem in (3.1), we have: (i) the map $v_0 \to v(t)$ defines a s.c. contraction semigroup on $L_2(\Omega)$: $v(t) = e^{L_F t}v_0 \in C([0,T]; L_2(\Omega))$, where L_F is a maximal dissipative operator, explicitly defined in [L-T-Z.2], [L-T.9]; (ii) for $v_0 \in L_2(\Omega)$, we have $e^{L_F t}v_0 \to 0$ in $L_2(\Omega)$, as $t \to \infty$.*

(Uniform stabilization [L-T-Z.2, Thm. 11.1.2]) Assume that there exists a (i) coercive real-valued vector field $\ell(x) \in (C^2(\overline{\Omega}))^n$ *[that is, with Jacobian matrix* J *satisfying* $Re\{Jv \cdot \bar{v}\} \geq \rho |v|^2$, $\rho > 0$*; in particular,* $\ell(x) = \nabla d(x)$*, for a real strictly convex function* $d(x)$ *on* Ω*, the radial case* $\ell(x) = x - x_0$*, for some* $x_0 \in \mathbb{R}^n$*, being the canonical case], such that (ii)* $\ell \cdot \nu \leq 0$ *on* Γ_0*. Then, there exist constants* $M \geq 1$, $\delta > 0$*, such that*

$$\|e^{L_F t}\|_{\mathcal{L}(L_2(\Omega))} \leq M e^{-\delta t}, \quad t \geq 0; \ \textit{equivalently, } E_v(t) \leq M e^{-\delta t} E_v(0), \ t \geq 0. \quad (3.2)$$

The proof of *generation* and *strong stability* is by "soft" methods: Lumer-Phillips Theorem for generation and a combination of Stone's and Nagy-Foias-Foguel's results in contraction semigroups [Lev.1] for strong stability, following an established procedure [L-T.2], [L-T.3], [Tr.2], etc. This avenue requires that the resolvent of the generator be compact, a property presently satisfied [L-T-Z.2, Sect. 11]. An alternative avenue consists of invoking the characterization of strong stability in [A-B.1], [L-P.1], which does not require that the generator has compact resolvent. In sharp contrast the *uniform stabilization* result relies on the non-trivial, general, *a-priori* energy estimate (1.8) of Theorem 1.1, a much harder avenue.

Remark 3.2. Via a well-known result of [Ru.1], Theorem 3.1 implies exact controllability in the state space $L_2(\Omega)$ in the sense of Theorem 2.1, at least for the y-problem on the LHS of (3.1) with Neumann boundary control u. Thus, Theorem 2.1 refers to a more general model (ultimately in the Riemannian setting of Remark 2.1 [Tr.3]).

3.2. Nonlinear boundary dissipation. Assumptions; main results. **Nonlinear boundary dissipation model.** Let Ω be an open bounded domain of $\mathbb{R}^n$, $n \geq 1$, as in Section 3.1, with sufficiently smooth boundary $\partial\Omega = \Gamma = \overline{\Gamma_0 \cup \Gamma_1}$, $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$, $\Gamma_0 \neq \emptyset$. Prompted by the dissipative linear v-problem in (3.1a–c) reported in Section 3.1, we now consider the corresponding problem with nonlinear boundary dissipation:

$$\begin{cases} iw_t + \Delta w = 0 & \text{in } Q = (0, T] \times \Omega; & (3.3a) \\ w(0, \cdot\,) = w_0 & \text{in } \Omega; & (3.3b) \\ w|_{\Gamma_0} \equiv 0, \ \dfrac{\partial w}{\partial \nu} = ig(w) & \text{in } \Sigma_k = (0, T] \times \Gamma_k, \ k = 0, 1. & (3.3c) \end{cases}$$

We have already noted in Remark 3.1 that we could take $\Gamma_0 = \emptyset$, and, moreover, we could take $\frac{\partial w}{\partial \nu}|_{\Gamma_0} \equiv 0$ instead of $w|_{\Gamma_0} \equiv 0$ in (3.2c), in which case the condition $\Gamma_0 \cap \Gamma_1 = \emptyset$ is dispensed with. However, stronger geometrical conditions, noted in Remark 3.1, are then called for.

Assumptions on the nonlinearity g. First, for purposes of the well-posedness result, Theorem 3.2 below, we impose on the complex-valued function g the following assumptions:

(H.1): g is a continuous complex, single-valued function of the complex variable $z \in \mathbb{C}$, $g(z)$, with $g(0) = 0$; moreover, $g(z)$ is the sub-differential, $g(z) = \partial j(z)$, of a lower semicontinuous, convex, proper function $j : \mathbb{C} \to \overline{\mathbb{R}} =]-\infty, +\infty]$, so that it satisfies

$$\operatorname{Im}\{g(z)\bar{z}\} \equiv 0; \quad \operatorname{Re}\{(g(z) - g(v))(\bar{z} - \bar{v})\} \geq 0, \quad \forall\, z, v \in \mathbb{C}. \tag{3.4a}$$

Thus, in particular, for $v = 0$ we obtain $\operatorname{Re}\{g(z)\bar{z}\} \geq 0$, so that

$$\operatorname{Re}\{g(z)\bar{z}\} = g(z)\bar{z} = |g(z)\bar{z}| \geq 0, \quad \forall\, z \in \mathbb{C}. \tag{3.4b}$$

Property (3.4a) is the present counterpart of the monotonicity assumption in the case g is real-valued function of a real variable. We may view $g : \mathbb{C} \to \mathbb{C}$ with the range space $\mathbb{C}$ being an inner-product space under the inner product $(z, v)_{\mathbb{C}} = \operatorname{Re}\{z\bar{v}\}$, $\forall z, v \in \mathbb{C}$. In the linear case of Section 3.1, we have $g(z) = z$. If the functions $g_j(z)$ satisfy (H.1), so does $\sum_{j=1}^{J} d_j g_j(z)$ for $d_j \geq 0$. Typical examples for $g(z)$ are: $g(z) = |z|^r z$, $r > 0$; or $g(z) = \frac{1}{|z|^r} z$, $0 < r < 1$. See Lemma 3.4; Examples #1–#4; [L-T.9, Appendix A] in Section 3.5.

Second, for purposes of the main result, the uniform stabilization Theorem 3.3 below, we require—besides (H.1)—additional growth conditions:

(H.2): There exist positive constants $m > 0$, $M > 0$, such that

(a)

$$m|z|^2 \leq g(z)\bar{z} \text{ (recall (3.4b)), for } |z| \geq 1, \quad \forall\, z \in \mathbb{C}; \tag{3.5a}$$

(b)

$$\begin{cases} |g(z)| \leq M|z|^p, \text{ for } |z| \geq 1, \quad \forall\, z \in \mathbb{C}; \\ \text{where: } p = 5 \quad \text{for } n = \dim \Omega = 2, \\ \qquad\quad\;\; p = 3 \quad \text{for } n = \dim \Omega = 3. \end{cases} \tag{3.5b}$$

We remark that *no growth assumptions on $g(z)$ are made near the origin*, though the decay asserted by Theorem 3.3 does depend on such behavior. Moreover, $g(\,\cdot\,)$ is allowed to be superlinear at infinity, unlike the corresponding case of the wave equation [La-Ta.1]. Finally, in contrast with most of the literature on uniform stabilization of nonlinear dynamics, g need not be differentiable. Classes of functions satisfying assumptions (H.1), (H.2) are discussed below in Lemma 3.4; [L-T.9, Appendix A], and Examples #1–#4 in Section 3.5.

Geometrical assumption. Finally, for purposes of the uniform stabilization Theorem 3.3, we need a geometrical condition imposed on the triple $\{\Omega, \Gamma_0, \Gamma_1\}$, $\Gamma_0 \neq \emptyset$, the same as in Theorem 3.1 and Theorem 2.1 of Section 2.1: there exists a real coercive vector field $\ell(x) \in (C^2(\overline{\Omega}))^n$, such that

(H.3))

$$\ell(x) \cdot \nu(x) \leq 0 \text{ on } \Gamma_0. \tag{3.6}$$

Main results. Well-posedness and regularity.

Theorem 3.2. *Let $n = 1, 2, \ldots$. Assume hypothesis (H.1) = (3.4) on g. Then, the following results hold true for the w-problem (3.3):*

(a) (well-posedness) For any initial condition $w_0 \in L_2(\Omega)$, problem (3.3) defines a unique (non-linear contraction semigroup) mild solution $w(t; w_0)$ [B.1, p. 202, 204, 230] satisfying

$$w_0 \in L_2(\Omega) \Rightarrow w(\,\cdot\,; w_0) \in C([0, \infty); L_2(\Omega)) \tag{3.7}$$

continuously. The generator A_F of the corresponding nonlinear semigroup is given explicitly in [L-T.9, Eqn. (4.2)]: it is maximal dissipative, hence closed; moreover, $\overline{\mathcal{D}(A_F)} = L_2(\Omega)$.
(b) (regularity) Let, in particular, $w_0 \in H^2(\Omega)$ subject to compatibility conditions:

$$w_0 \in H^2(\Omega) : w_0|_{\Gamma_0} = 0; \ \left.\frac{\partial w_0}{\partial \nu}\right|_{\Gamma_1} = ig(w_0), \quad \text{so that } w_0 \in \mathcal{D}(A_F), \tag{3.8a}$$

the domain of the generator A_F defined in [L-T.9, Eqn. (4.2)]. Then, the corresponding unique solution $w(t; w_0)$ guaranteed by part (a), satisfies [B.1, Thm. 1.2, p. 220] (w_t^+ = right-derivative)

$$\begin{cases} w(\,\cdot\,; w_0) \in C([0, \infty); \mathcal{D}(A_F)), \mathcal{D}(A_F) \subset \mathcal{D}(A^{\frac{1}{2}}) \equiv H^1_{\Gamma_0}(\Omega), w_t^+ \in C([0, \infty); L_2(\Omega)); \\ w(\,\cdot\,; w_0)|_{\Gamma_1} \in C([0, \infty); H^{\frac{1}{2}}(\Gamma_1)). \end{cases} \tag{3.8b}$$

(c) (higher regularity) Assume (3.8a) on w_0 and, moreover,
(c_1) if dim $\Omega = 2$, assume that $g(z)$ is of polynomial growth:

$$|g(z)| \le C|z|^k, \quad |z| \ge 1, \text{ for some positive integer } k, \tag{3.9a}$$

or, more generally, that $g: H^{\frac{1}{2}}(\Gamma) \to L_2(\Gamma)$, in particular, $g: L_p(\Gamma) \to L_2(\Gamma)$, $p \ge 1$;
(c_2) if dim $\Omega = 3$, assume that

$$|g(z)| \le C|z|^r, \quad z \in \mathbb{C}, \text{ for some } r < 3 \tag{3.9b}$$

($r = 3 - \epsilon$, $\epsilon > 0$ arbitrary). Then, in both cases (c_1) and (c_2), we have that

$$\mathcal{D}(A_F) \subset H^{\frac{3}{2}}(\Omega), \text{ so that } w(\,\cdot\,; w_0) \in C([0, \infty); H^{\frac{3}{2}}(\Omega)), \tag{3.10a}$$

for w_0 as in (3.8a). In particular (from (3.8b) and (3.9a–b)):

$$\left.\frac{\partial w}{\partial \nu}\right|_{\Gamma_1} = ig(w(\,\cdot\,; w_0)|_{\Gamma_1}) \in L_2(0, T; L_2(\Gamma_1)). \tag{3.10b}$$

Additionally, higher regularity requires differentiability of $g(\,\cdot\,)$. We shall not include this result. Of course, Theorem 3.2 is a generalization of the linear well-posedness Theorem 3.1(i) to which it reduces for $g(z) = z$. Theorem 3.2 follows from monotone operator theory. Its proof is given in [L-T.9, Appendix C].

The main focus of the present section is an asymptotic energy decay rates of solutions, as $t \to \infty$. To this end, assumptions (H.2) on g and (H.3) on geometrical conditions are invoked.

Uniform stabilization (or uniform decay rates). Orientation. Before stating our main result, Theorem 3.3, we need to introduce some concepts and relative notation. The proof, given in [L-T.9, Section 5], needs to establish some *a-priori* estimates, after which one is able to fall into the general setting for "hyperbolic-like" linear PDE-dynamics (wave, Schrödinger, plates, shells equations) with nonlinear (interior and) boundary dissipation, first introduced in [La-Ta.1] in the specific case of wave equations with nonlinear boundary dissipation in the Neumann-boundary conditions. This approach is, however, fully general. It has been used in several other settings, including the following cases: von Karman plates [H-L.1]; full von Karman model [Las.1]; Maxwell equations [E-L-N.1]; equations of shells (coupled system of two hyperbolic PDEs defined on a 2-dimensional surface) [L-T.4]; wave equations with interior localized dissipation [La-To.1]. It will be invoked and applied also in the present setting of the Schrödinger problem (3.3). We need to recall such strategy.

Step (i): The concave function $h(x)$**.** Following [La-Ta.1], we let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a real-valued continuous, concave, strictly increasing function, with $h(0) = 0$, satisfying

$$h(g(z)\bar{z}) \geq |z|^2 + |g(z)|^2, \text{ for } |z| \leq \delta, \text{ for some } \delta > 0,\ z \in \mathbb{C} \tag{3.11}$$

(recall that $g(z)\bar{z} \geq 0$ by (3.4b)). Such function $h(s)$, $s \geq 0$, can always be constructed on the strength of assumptions (H.1), (H.2). Thus, (3.11) is a property, not an assumption.

Having constructed the required function $h(\,\cdot\,)$, we rescale it by letting

$$\tilde{h}(x) = h\left(\frac{x}{\text{meas}(\Sigma_1)}\right), \quad x \geq 0, \tag{3.12}$$

where $\Sigma_1 = (0,T]\times\Gamma_1$, with $T > 0$ arbitrary, and "meas" is the cylinder Q's surface measure. Of course, $\tilde{h}(0) = 0$ and $\tilde{h}(x)$ is strictly increasing, so that the operator $CI + \tilde{h}$ is invertible for any constant $C \geq 0$, where I is the identity operator.

Step (ii): The convex function $p(x)$**.** For any positive constant C, whereby the concave function $(C + \tilde{h})$ is invertible, and for any positive constant K, we define the convex function $p(x)$ by setting

$$p(x) = (CI{+}\tilde{h})^{-1}(Kx) : \text{positive for } x > 0,\ p(0) = 0, \text{ continuous, strictly increasing.} \tag{3.13}$$

Thus, the function $(I + p)(x)$ is invertible. [The present function $p(x)$ should not be confused with the parameter p in (3.5b).]

Step (iii): The function $q(x)$**.** Finally, we define the function $q(x)$ by

$$\begin{aligned} q(x) \quad &= \quad x - (I+p)^{-1}(x) = p(I+p)^{-1}(x) = (I+p)^{-1}p(x) \\ &\quad : \text{ positive for } x > 0,\ q(0) = 0, \text{ continuous, strictly increasing.} \end{aligned} \tag{3.14}$$

[Differentiating $q(x) + p(q(x)) = p(x)$ yields $q'(x) = p'(x)/[1 + p'(q(x))] > 0$, since $p(x)$ is strictly increasing. Also, $[I + p]q(0) = p(0) = 0 \Rightarrow q(0) = 0$.] We note that the above procedure—step (i) through step (iii) is both *constructive* and *explicit*, given the data of the problem: the nonlinear function g satisfying assumption (H.1); the part Γ_1 of the boundary, and the constant $T > 0$. See [L-T.9, Section 3].

We can now state our main uniform decay rate result for problem (3.3).

Theorem 3.3. *Let $n = \dim \Omega = 2, 3$. With reference to the w-problem (3.3), we assume hypotheses (H.1) = (3.4), (H.2) = (3.5) for g and (H.3) = (3.6) for $\{\Omega, \Gamma_0, \Gamma_1\}$. Then, the energy*

$$E(t) \equiv \|w(t, w_0)\|^2_{L_2(\Omega)}, \quad w_0 \in L_2(\Omega) \tag{3.15}$$

of the solution w of problem (3.3), guaranteed by Theorem 3.2, satisfies the following decay rate

$$E(t) \le S\left(\frac{t}{T_0} - 1\right)(E(0)) \searrow 0 \quad \textit{for all } t \ge T_0,\ t \to \infty, \tag{3.16}$$

for some $T_0 > 0$, where the scalar function $S(t)$ (nonlinear contraction) is the solution of the following nonlinear ODE:

$$\frac{d}{dt}S(t) + q(S(t)) = 0, \quad S(0) = E(0) \equiv \|w_0\|^2_{L_2(\Omega)}, \tag{3.17}$$

where the function q is defined by (3.14), via (3.13), where the positive constants C and K there are defined by

$$C = \frac{\frac{1}{m} + 1 + \tilde{C}_p(E(0))^{\frac{p-1}{p+1}}}{\operatorname{meas}(\Sigma_1)}; \qquad K = \frac{1}{2C_T\ \operatorname{meas}(\Sigma_1)}. \tag{3.18}$$

where $p = 5$ for $\dim\,\Omega = 2$, $p = 3$ for $\dim\,\Omega = 3$, see (3.5b); m is defined in (3.5a); $\tilde{C}_p$ is given explicitly in [L-T.9, Proposition 5.2.1]; and C_T is the constant in [L-T.9, (5.1.3)]. Thus, from (3.16) it follows that

$$E(t) \to 0 \textit{ as } t \to \infty, \textit{ with rates specified by } S(t). \tag{3.19}$$

Theorem 3.3 generalizes the linear Theorem 3.1. We remark that the function q (like p) depends on the constants C and K. In the stabilization result of Theorem 3.3 above, C depends, in turn, on the data as well as on $E(0)$, the initial energy. Thus, the decay provided by Theorem 3.3 is uniform with respect to all initial conditions within a same ball of $L_2(\Omega)$, centered at the origin. Several illustrations on the application of Theorem 3.3, for various functions g and corresponding rates are given in [L-T.9, Section 3]; and a few more are given in Section 3.3 below.

Preliminary dissipation energy identity. Once the well-posedness result [L-T.9, Theorem 3.2(b)] has been established—in particular (3.8b) for w_t, $\nabla w \in L_2(\Omega)$ a.e., and $\frac{\partial w}{\partial \nu}|_{\Gamma_1} \in L_2(\Gamma_1)$ a.e. for strong solutions—the following standard energy method is justified. We multiply Eqn. (3.3a), rewritten equivalently as $w_t = i\Delta w$ in Q, by $\bar{w}$, take real parts of the resulting identity and integrate by parts in time and space over $\int_s^t \int_\Omega d\Omega\, d\tau$. We use $\frac{\partial}{\partial t}|w|^2 = 2\,\mathrm{Re}\{w_t \bar{w}\}$ in the integration in t on the left side, and Green first theorem in the space integration on the right side with boundary conditions given by (3.3c). In the process, we get the cancelation $\mathrm{Re}(i|\nabla w|^2) = 0$, and finally obtain the identity

$$\int_\Omega |w(t)|^2 d\Omega + 2\int_s^t \int_{\Gamma_1} \mathrm{Re}\{g(w)\bar{w}\} d\Gamma_1 d\tau = \int_\Omega |w(s)|^2 d\Omega,\ t \geq s \geq 0. \tag{3.20a}$$

Invoking property (3.4b) on g, $\mathrm{Re}\{g(w)\bar{w}\} = g(w)\bar{w} \geq 0$, as well as the energy $E(\,\cdot\,)$ in (3.15), we can rewrite identity (3.20a) as follows, so that the energy is monotonically decreasing, as desired:

$$E(t)+2\int_s^t \int_{\Gamma_1} g(w)\bar{w}\, d\Gamma_1 d\tau = E(s),\ \text{where then } E(t) \leq E(s),\ t \geq s \geq 0, \tag{3.20b}$$

Remark 3.3. The energy method leading to identity (3.20a) gives $\mathrm{Re}\{g(w)\bar{w}\}$ in the boundary integral. On the other hand, the boundary integral in the critical estimate (1.8) of Section 1, when applied with $z = w$, the solution of problem (3.3), gives the integrand $|g(w)|\,|w|$: see [L-T.9, (5.1.1)]. It is the need to match these two integrands: $\mathrm{Re}\{g(w)\bar{w}\} = |g(w)\bar{w}|$, that forces (part of) assumption (H.4) = (3.4a-b). As a result, it is precisely in the form (3.20b) that one invokes the dissipativity identity (3.20a). See e.g., [L-T.9, (5.2.29) via (5.2.1)]. $\square$

A first direct construction of continuous functions $g(z)$ satisfying (3.4a) near the origin.

Lemma 3.4. *With $s_0 > 0$, let*

$\gamma : [0, s_0] \to \mathbb{R}_+$ be a continuous function of a real variable, $\gamma(0) = \lim_{s\downarrow 0} \gamma(s) \geq 0$, $\gamma(s) > 0$ for $s > 0$, such that $s \to s\gamma(s)$ is monotone increasing, (3.21)

the case $\gamma(0) = +\infty$ being included. Define the continuous function $g(z)$ by

$$g(z) = \gamma(|z|)z, \quad z \in \mathbb{C} \text{ and assume that } g(0) = 0, \tag{3.22}$$

so that $g(s)$ is increasing on $[0, s_0]$. Then $g(z)$ in (3.22) satisfies assumption (3.4a).
(b) As a partial converse, let $g(z)$ be a continuous function of $z \in \mathbb{C}$. Then, with reference to (3.4a) (LHS), we have

$$Im\{g(z)\bar{z}\} \equiv 0,\ \forall\ z \in \mathbb{C} \quad \Rightarrow g(z) = f(x, y)z, \tag{3.23}$$

where $f(x, y)$ is a continuous, real-valued function of the real variables $x = Re\ z$, $y = Im\ z$, $z = x + iy$. Moreover, with reference to (3.4b) we have

$$Re\{g(z)\bar{z}\} \geq 0,\ \forall\ z \in \mathbb{C} \quad \Rightarrow f(x, y) \geq 0,\ \forall\ x, y \in \mathbb{R}. \tag{3.24}$$

Proof of Lemma 3.4. (a) Property (3.4b) is obvious: $g(z)\bar{z} = \gamma(|z|)|z|^2 \geq 0$, $z \in \mathbb{C}$, while $g(0) = 0$. As to property (3.4a), we compute from (3.4b), for $z, v \in \mathbb{C}$, via

(3.21), (3.22):

$$\begin{aligned}
\mathrm{Re}\{(g(z)-g(v))(\bar z-\bar v)\} &= \mathrm{Re}\{[\gamma(|z|)z-\gamma(|v|)v](\bar z-\bar v)\}\\
&= \gamma(|z|)|z|^2+\gamma(|v|)|v|^2-\gamma(|z|)\,\mathrm{Re}\{z\bar v\}-\gamma(|v|)\,\mathrm{Re}\{v\bar z\}\\
&= \gamma(|z|)|z|^2+\gamma(|v|)|v|^2-[\gamma(|z|)+\gamma(|v|)]\,\mathrm{Re}\{z\bar v\} \qquad (3.25)\\
&\geq \gamma(|z|)|z|^2+\gamma(|v|)|v|^2-[\gamma(|z|)+\gamma(|v|)]|z|\,|v|\\
&= \gamma(|z|)|z|^2+\gamma(|v|)|v|^2-\gamma(|z|)|z|\,|v|-\gamma(|v|)|z|\,|v|\\
&= \gamma(|z|)|z|[|z|-|v|]+\gamma(|v|)|v|[|v|-|z|]\\
&= (\gamma(|z|)|z|-\gamma(|v|)|v|)(|z|-|v|)\\
&= (\gamma(r)r-\gamma(\rho)\rho)(r-\rho)=(g(r)-g(\rho))(r-\rho)\geq 0, \quad \forall\, r,\rho>0, \qquad (3.26)
\end{aligned}$$

where we have set $|z|=r$, $|v|=\rho$, and where non-negativity follows since the function $r\to r\gamma(r)$ is increasing. Thus, (3.26) proves (3.4a), as desired.
(b) Let $z=x+iy$, $x,y\in\mathbb{R}$ and write $g(z)=u(x,y)+i\mu(x,y)$, $u,\mu\in\mathbb{R}$. We have $g(z)\bar z=(ux+\mu y)+i(x\mu-uy)$. The condition $\mathrm{Im}\{g(z)\bar z\}\equiv 0$ in (3.23) (LHS) implies: $x\mu=uy$, or $u=f(x,y)x$, $\mu=f(x,y)y$, for some continuous, real-valued function $f(x,y)$. Hence, $g(z)$ must be of the form $g(z)=f(x,y)(x+iy)=f(x,y)z$, and property (3.23) (RHS) is established. Property (3.24) is then obvious. $\square$

The above proof is then integrated in the general abstract setting given in [L-T.9, Appendix A, point (5)], to obtain further that $g(z)=\partial_j(z)$, as required by the full assumption (H.1). A canonical distinctive class of functions $\gamma(s)$ covered by Lemma 3.4 is given by

$$\gamma(s)=s^r,\ \text{hence } g(z)=|z|^r z,\ \text{for all real } r>0, \qquad (3.27)$$

since $s\gamma(s)=s^{r+1}$ is monotone increasing, $s>0$. This class is also noted in [Lio.2, p. 133].

The *canonical* cases to distinguish are three:
Case 1 (fast decay to zero: superlinear):

$$\gamma(s)=s;\ g(s)=s^2,\ 0\leq s\leq s_0;\ g(z)=|z|z,\ 0\leq|z|\leq s_0. \qquad (3.28)$$

Case 2 (linear case):

$$\gamma(s)\equiv 1,\ g(s)=s,\ 0\leq s\leq s_0;\ g(z)=z,\quad 0<|z|\leq s_0. \qquad (3.29)$$

Case 3 (slow decay to zero: sublinear):

$$\gamma(s)=\frac{1}{\sqrt{s}},\ g(s)=\sqrt{s},\ 0<s\leq s_0;\ g(z)=\frac{1}{\sqrt{|z|}}\,z,\ 0<|z|\leq s_0. \qquad (3.30)$$

The two corollaries in Section 3.3 below deal with general classes of which Case 1 and Case 3 are, respectively, canonical representatives. Further illustrations are given in Section 3.3 below, complementing those given in [L-T.9, Section 3].

Case 1 and Case 3 correspond to slow energy decay rates (polynomial type or worse). Case 2 corresponds to exponential energy decay rate. See Examples #1–#4 at the end of Section 3.5.

Literature on boundary feedback stabilization of Schrödinger equations. Linear case. This is reviewed in [L-T-Z.2, Section 11]. In a nutshell: [L-T.3] for uniform decay in the space of optimal regularity $H^{-1}(\Omega)$ with Dirichlet (non-local) dissipativity and [M.1] for uniform decay in the (excessively smooth) space $H^1(\Omega)$ with Neumann dissipation involving $v_t \sim \Delta v$ both in the linear case. Compare with Theorem 3.1 in the more desirable space $L_2(\Omega)$ with the more desirable Neumann dissipation involving v_i in the linear case; and also with Theorem 3.3, the nonlinear case.

Nonlinear case. Contributions of the paper [L-T.9], here reviewed. In short, the main novel features of our paper include the following points:

Well-posedness. (i) A new abstract well-posedness result—[L-T.9, Theorem C.1.1 in Appendix C], specialized next to the case of the Schrödinger equation with nonlinear monotone interior and boundary damping in [L-T.9, Theorem C.2.1] and, further, in [L-T.9, Theorem 3.2]. Such result requires a special proof within the theory of maximal monotone operators, which is definitely more challenging than in the corresponding case of the wave equation (see [L-T.9, Remark C.1.1]), as to necessitate the approximation argument of [L-T.9, Section C.1].

Uniform stabilization. (ii) sharp (optimal) energy decay rates, under a nonlinear, attractive boundary dissipation in the Neumann B.C., are obtained in the desirable $L_2(\Omega)$-norm in Theorem 3.3 with four illustrative computed examples, see below.

(iii) The non-dissipative (homogeneous or unobserved) part Γ_0 of the boundary is allowed also to be of Neumann-type (case where the Lopatinski Condition is not satisfied), as noted in Remark 3.1: in this case, the price to pay is a stronger geometrical condition (after [L-T-Z.1, Appx. A] in the wave equation case, e.g., Γ_0 convex or concave); at any rate, $\overline{\Gamma}_0$ and $\overline{\Gamma}_1$ need not be disjoint.

(iv) Superlinear growth of the boundary dissipation is allowed at infinity: up to polynomial growth of order 5, for dim $\Omega = 2$; and of order 3, for dim $\Omega = 3$. By contrast, in the case of the wave equation, [La-Ta.1] allowed only linear growth at infinity for the boundary monotone damping. Superlinear growth in the Schrödinger's case is the result of Carleman's estimates penalizing normal derivatives on the boundary in negative anisotropic norms (Theorem 1.1 of Section 1).

(v) No growth assumption on the (monotone) nonlinearity is required at the origin, which therefore may be arbitrary. However, the decay rates that correspondingly are obtained via a constructive algorithm (a refinement of [La-Ta.1]) and are entirely determined by the behavior of the (monotone) nonlinearity at the origin (see the numerous examples in Section 3.3). In particular, in contrast with most of the literature on uniform stabilization of nonlinear dynamics, no differentiability of the dissipation is assumed.

Regarding (iv), a recent analysis of the wave equation with nonlinear localized interior damping and source terms and no growth restrictions at infinity is carried out in [La-To.1]. In both the linear and nonlinear Schrödinger cases, the underlying supporting pillar for obtaining decay rates in the $L_2(\Omega)$-topology is a Carleman-type energy estimate at the $L_2(\Omega)$-level (Theorem 1.1). It is obtained by use of a pseudo-differential shift of topology, and related micro-local analysis [L-T-Z.2, Sect. 10] (see also [L-T-Z.2]) to go from the natural $H^1(\Omega)$-level to the desired $L_2(\Omega)$-level. We also refer to [Ta.1] in the Dirichlet case.

3.3. Corollaries and illustrations: computation of optimal decay rates. In this section, in order to illustrate Theorem 3.3, we refer to two general corollaries [L-T.9, Section 3]. They refer to the 'end cases' where the function $g(s)$, in the real positive variable s, has either a *fast decay* as in Case 1, (3.28), ([L-T.9, Corollary 3.5]) or else a *slow decay* as in Case 3, (3.30) [L-T.9, Corollary 3.6] *to zero* near $s = 0$, where we recall that $g(0) = 0$. Next, we present a few significant illustrations of these two corollaries. In each of them, we compute explicitly the rates of decay through the function q in (3.14)) which are, in fact, optimal. These are obtained through the explicit sequential algorithmic procedure that was introduced in [La-Ta.1] and is reproduced here—as adapted to the present class of complex-valued boundary terms $g(z)$ in (3.22). It consists in the successive construction of the following array of functions: $g \to h \to p \to q$, starting from the given boundary term $g(z) = \gamma(|z|)z$, as in (3.22).

Corollary 3.5. *(fast decay to zero of $g(s)$ as $s \downarrow 0$) Assume (H.1) = (3.4), (H.2) = (3.5), (H.3) = (3.6).*

(a) Let $\gamma(s)$ be the function: $[0, s_0] \to \mathbb{R}_+$ defined by (3.21), and let then $g(s) = s\gamma(s)$ monotone increasing, $g(0) = 0$, $g(z) = \gamma(|z|)z$ as in (3.22). Assume further that

(a_1) $0 \le \gamma(0) < 1$;

(a_2) the function $s\gamma(\sqrt{s}) = \sqrt{s}\, g(\sqrt{s})$ is convex near $s = 0$, say for $0 < s < s_0^2$, for some s_0.

Then, the procedure described in Step (i), Eqn. (3.11), to construct the required continuous, concave, strictly increasing function $h(x) : \mathbb{R}_+ \to \mathbb{R}_+$, with $h(0) = 0$, can be arranged as to yield the following choice

$$h^{-1}(x) = \frac{x}{2}\gamma\left(\sqrt{\frac{x}{2}}\right) = \sqrt{\frac{x}{2}}\, g\left(\sqrt{\frac{x}{2}}\right), \text{near } x = 0, \text{ say, } 0 \le x \le x_0 = 2s_0^2. \quad (3.31)$$

(b) Let $C > 0$ be the constant in the definition of the sought-after continuous, strictly increasing function $p(x)$, $p(0) = 0$, of Step (ii) in (3.13). For illustration purposes, assume that meas$(\Sigma_1) = 1$, so that $\tilde{h} \equiv h$, see (3.12). Assume further that

$$\gamma(s) \text{ is } C^1 \text{ for } s \downarrow 0, \text{ and that } \quad \frac{1}{4}[\gamma(0) + g'(0)] < \frac{1}{C}, \quad (3.32)$$

where $g'(0)$ must then be $g'(0) \ge 0$. Then, we can arrange to take the function $p(x)$ as to satisfy

$$h^{-1}\left(\frac{x}{2}\right) \le p(x) \le h^{-1}(x) = \frac{x}{2}\gamma\left(\sqrt{\frac{x}{2}}\right) = \sqrt{\frac{x}{2}}g\left(\sqrt{\frac{x}{2}}\right) < \frac{x}{2}, 0 < x \le x_1, x_1 \text{ small.} \quad (3.33)$$

Also, under assumptions (a), (b), we can always arrange to have the function $q(x)$ in (3.14) satisfy for x near $x = 0$:

$$\tilde{q}(x) \equiv \frac{2}{3}h^{-1}\left(\frac{x}{2}\right) \leq \frac{2}{3}p(x) \leq q(x) \leq p(x) \leq h^{-1}(x) < \frac{x}{2}; 0 \leq x \leq x_2; \tag{3.34a}$$

$$h^{-1}\left(\tfrac{x}{2}\right) = \frac{x}{4}\gamma\left(\sqrt{\frac{x}{4}}\right) = \sqrt{\frac{x}{4}}\,g\left(\sqrt{\frac{x}{4}}\right), 0 < x \leq x_3 = \textit{small}. \tag{3.34b}$$

The function $\tilde{q}(x)$ defined in (3.34) satisfies: $\tilde{q}(0) = 0$, $\tilde{q}(x) > 0$ for $0 < x \leq x_3$, and $\tilde{q}(x)$ is strictly increasing and convex. Consider the new ODE

$$\tilde{S}_t(t) + \tilde{q}(\tilde{S}(t)) \equiv 0, \tag{3.35a}$$

or

$$\tilde{S}_t(t) + \frac{2}{3}\,\frac{\tilde{S}(t)}{4}\gamma\left(\sqrt{\frac{\tilde{S}(t)}{4}}\right) = \tilde{S}_t(t) + \frac{2}{3}\sqrt{\frac{\tilde{S}(t)}{4}}\,g\left(\sqrt{\frac{\tilde{S}(t)}{4}}\right) = 0, \tilde{S}(T) = S(T), \tag{3.35b}$$

where T is sufficiently large, so that the solution $S(t)$ of the ODE (3.17) evaluated at $t = T$ satisfies $S(T) < x_3$ (this is possible by (3.16)). Then, the solutions of the corresponding w-problem (3.3) with such $g(z) = \gamma(|z|)z$, for $|z|$ small [and otherwise subject to (H.1) = (3.4) and (H.2) = (3.5)] satisfy

$$E(t) \leq C(E(0))S(t) \leq C(E(0))\tilde{S}(t),\ t > T \text{ and } \tilde{S}(t) \searrow 0 \text{ as } t \to \infty, \tag{3.36}$$

([L-T.9, (B.4)]) where $\tilde{S}(t)$ is obtained from integrating

$$8\int_{\sqrt{\frac{\tilde{S}(T)}{4}}}^{\sqrt{\frac{\tilde{S}(t)}{4}}} \frac{du}{g(u)} = \frac{2}{3}(T - t),\ u = \sqrt{\frac{\tilde{S}}{4}}. \tag{3.37}$$

Corollary 3.6. *(slow decay to zero of $g(s)$ as $s \downarrow 0$) Assume (H.1) = (3.4), (H.2) = (3.5), (H.3) = (3.6).*

(a) Let $\gamma(s)$ be the function: $(0, s_0] \to \mathbb{R}_+$ defined by (3.21), and let then $g(s) = s\gamma(s)$ monotone increasing, $g(0) = 0$, $g(z) = \gamma(|z|)z$ as in (3.22). Assume further that

(a_1) $\gamma(0) > 1$; thus, $\lim_{s\searrow 0}\frac{s}{g(s)} = \lim_{s\searrow 0}\frac{1}{\gamma(s)} = \frac{1}{\gamma(0)} \leq 1$; where the case $\gamma(0) = \lim_{s\downarrow 0}\gamma(s) = +\infty$ is included (and is typical of Case 3, (3.30)).

(a_2) the function $g(s)$, in the real positive variable s, is (not only increasing, as contained in (3.4a) via Lemma 3.4 for z and v restricted to real positive variables, but also) strictly increasing near $s = 0$, with inverse $g^{-1}(\cdot)$. Moreover, the function $\sqrt{s}\,g^{-1}(\sqrt{s})$ is convex near $s = 0$.

Then, the procedure described in Step (i), Eqn. (3.11), to construct the required continuous, concave, strictly increasing function $h(x) : \mathbb{R}_+ \to \mathbb{R}_+$, with $h(0) = 0$, can be arranged as to yield the following choice

$$h^{-1}(x) = \sqrt{\frac{x}{2}}\,g^{-1}\left((\sqrt{\frac{x}{2}}\right) < \frac{x}{2},\ \text{near } x = 0,\ \text{say, } 0 \leq x \leq x_0 \text{ small.} \tag{3.38}$$

(b) Let $C > 0$ be the constant in the definition of the sought-after continuous, strictly increasing function $p(x)$, $p(0) = 0$, of Step (ii) in (3.13). For illustration purposes, assume that meas$(\Sigma_1) = 1$, *so that $\tilde{h} = h$, see (3.12). Assume further that*

$$\gamma(s) \text{ is } C^1 \text{ for } s \downarrow 0 \text{ and } \left[\frac{1}{g'(0)} + \frac{1}{\gamma(0)}\right]^{-1} > \frac{C}{4} \tag{3.39}$$

[where the cases $g'(0) = \gamma(0) = +\infty$ are included and indeed typical of Case 2, (3.29)]. Then, we can arrange to take the function $p(x)$ as to satisfy

$$h^{-1}\left(\frac{x}{2}\right) \leq p(x) \leq h^{-1}(x) = \sqrt{\frac{x}{2}}\, g^{-1}\left(\sqrt{\frac{x}{2}}\right) < \frac{x}{2}, \tag{3.40}$$

near $x = 0$, say $0 < x \leq x_1$, x_1 small. Also, under assumptions (a), (b), we can always arrange to have the function $q(x)$ in (3.13) satisfy

$$\tilde{q}(x) \equiv \frac{2}{3} h^{-1}\left(\frac{x}{2}\right) \leq \frac{2}{3} p(x) \leq q(x) \leq p(x) \leq h^{-1}(x) = \sqrt{\frac{x}{2}} g^{-1}\left(\sqrt{\frac{x}{2}}\right) < \frac{x}{2},$$

$$0 \leq x \leq x_3 = \text{small}, \tag{3.41a}$$

$$h^{-1}\left(\frac{x}{2}\right) = \sqrt{\frac{x}{4}}\, g^{-1}\left(\sqrt{\frac{x}{4}}\right), \quad x \text{ small}, \tag{3.41b}$$

so that the ODE (3.35a): $\tilde{S}_t(t) + \tilde{q}(\tilde{S}(t)) \equiv 0$ now takes the form via (3.41)

$$\tilde{S}_t(t) + \frac{2}{3}\sqrt{\frac{\tilde{S}(t)}{4}}\, g^{-1}\left(\sqrt{\frac{\tilde{S}(t)}{4}}\right) = 0, \quad \tilde{S}(T) = S(T), \tag{3.42}$$

where T is sufficiently large, so that the solution $S(t)$ of the ODE (3.17) evaluated at $t = T$ satisfies $S(T) < x_3$ (this is possible by [L-T.9, (A.3)] or (3.17)). Thus, the solutions of the corresponding w-problem (3.3) with such $g(z) = \gamma(|z|)z$ for $|z|$ small [and otherwise subject to (H.1) = (3.4) and (H.2) = (3.5)], satisfy

$$E(t) \leq C(E(0))S(t) \leq C(E(0))\tilde{S}(t), \quad \forall\, t \geq T; \text{ and } \tilde{S}(t) \searrow 0 \text{ as } t \to \infty, \tag{3.43}$$

([L-T.9, (A.4)]) where $\tilde{S}(t)$ is obtained from integrating

$$\int_{\tilde{S}(T)}^{\tilde{S}(t)} \frac{d\tilde{S}}{\sqrt{\frac{\tilde{S}}{4}}\, g^{-1}\left(\sqrt{\frac{\tilde{S}}{4}}\right)} = 8 \int_{\sqrt{\frac{\tilde{S}(T)}{4}}}^{\sqrt{\frac{\tilde{S}(t)}{4}}} \frac{du}{g^{-1}(u)} = \frac{2}{3}(T - t); \quad u = \sqrt{\frac{\tilde{S}}{4}}. \tag{3.44}$$

Example #1. In this illustration, we take the superlinear case for $g(s)$ as in (3.22):

$$\gamma(s) = s^r \text{ or } g(s) = s^{r+1}, r > 0; \text{ near the origin, } 0 \leq s \leq s_0;\ g(z) = \gamma(|z|)z = |z|^r z. \tag{3.45}$$

Application of Corollary 3.5. We have

$$\gamma(0)=0;\ s\gamma(\sqrt{s})=s^{\frac{r+2}{2}} \text{ convex near } s=0,\ s>0;\ g'(0)=0, \tag{3.46}$$

and thus assumptions (a$_1$), (a$_2$), (b) = (3.32) of Corollary 3.5 are fulfilled for *any* constant $C>0$. By [L-T.9, (3.13) and (3.46)], we have $h'(0)=\infty$, and by [L-T.9, (3.18)], we have $p'(0)=0$. Recalling the definition of $\tilde{q}(x)$ in (3.34a)

$$\tilde{q}(x)=\frac{2}{3}h^{-1}\left(\frac{x}{2}\right)=\frac{2}{3}\sqrt{\frac{x}{4}}\,g\left(\sqrt{\frac{x}{4}}\right)=\frac{2}{3}\left(\sqrt{\frac{x}{4}}\right)^{r+1},\ \text{near } x=0, \tag{3.47}$$

via (3.45). Then we need to integrate Eqn. (3.37) with $\tilde{S}(T)=S(T)$ to get via (3.45),

$$\frac{2}{3}(T-t)=8\int_{\sqrt{\frac{S(T)}{4}}}^{\sqrt{\frac{\tilde{S}(t)}{4}}}\frac{du}{g(u)}=8\int_{\sqrt{\frac{S(T)}{4}}}^{\sqrt{\frac{\tilde{S}(t)}{4}}}u^{-(r+1)}du=\frac{8}{r}\left[\left(\frac{S(T)}{4}\right)^{-\frac{r}{2}}-\left(\frac{\tilde{S}(t)}{4}\right)^{-\frac{r}{2}}\right]; \tag{3.48}$$

$$\left(\frac{\tilde{S}(t)}{4}\right)^{-\frac{r}{2}}=\left(\frac{S(T)}{4}\right)^{-\frac{r}{2}}+\frac{2}{3}r(t-T),\ \text{or}\ \frac{\tilde{S}(t)}{4}=\left[\frac{1}{\left(\frac{S(T)}{4}\right)^{-\frac{r}{2}}+\frac{2}{3}r(t-T)}\right]^{\frac{2}{r}},\forall\, t\geq T. \tag{3.49}$$

By (3.36) of Corollary 3.5, since $S(0)=E(0)>S(T)=\tilde{S}(T)$, as $S(t)\searrow 0$ by (3.16) or [L-T.9, (B.3)], we conclude via (3.49) that: with $g(z)=\gamma(|z|)z=|z|^r z,\ r>0$ by (3.45), for $|z|$ small, and otherwise subject to assumption (H.2), the solutions of the corresponding w-problem (3.3) satisfy the following energy decay

$$E(t)\leq C(E(0))\left[\left(\frac{E(0)}{4}\right)^{-\frac{r}{2}}+\frac{2}{3}r(t-T)\right]^{-\frac{2}{r}},\ t\geq T. \tag{3.50}$$

Example #2. In this illustration, we take

$$\gamma(s)=se^{-\frac{1}{s}},\ \text{or}\ g(s)=s^2e^{-\frac{1}{s}} \text{ near } s=0,\ 0\leq s\leq s_0;\ g(z)=\gamma(|z|)z=|z|e^{-\frac{1}{|z|}}z. \tag{3.51}$$

Application of Corollary 3.5. We have

$$\gamma(0)=0;\ s\gamma(\sqrt{s})=s^{\frac{3}{2}}e^{-\frac{1}{\sqrt{s}}} \text{ convex near } s=0,\ g'(0)=0, \tag{3.52}$$

and thus assumptions (a$_1$), (a$_2$), (b) = (3.32) of Corollary 3.5 are fulfilled, again with *any* constant $C>0$. By [L-T.9, (3.13) and (3.52)], we have $h'(0)=\infty$, and hence by [L-T.9, (3.18)], we have $p'(0) = 0$. The definitions of $\tilde{q}(x)$ in (3.34) near $x = 0$ is, via (3.51):

$$\tilde{q}(x) = \frac{2}{3} h^{-1}\left(\frac{x}{2}\right) = \frac{2}{3}\sqrt{\frac{x}{2}}\, g\left(\sqrt{\frac{x}{2}}\right) = \frac{2}{3}\left(\frac{x}{2}\right)^{\frac{3}{2}} \exp\left(-\frac{1}{\sqrt{\frac{x}{2}}}\right). \tag{3.53}$$

Then we need to integrate Eqn. (3.37) with $\tilde{S}(T) = S(T)$ to get

$$\frac{2(T-t)}{3} = \int_{\sqrt{\frac{S(T)}{4}}}^{\sqrt{\frac{\tilde{S}(t)}{4}}} \frac{du}{g(u)} = \int_{\sqrt{\frac{S(T)}{4}}}^{\sqrt{\frac{\tilde{S}(t)}{4}}} \frac{e^{\frac{1}{u}}}{u^2} du = -\int_{\sqrt{\frac{4}{S(T)}}}^{\sqrt{\frac{4}{\tilde{S}(t)}}} e^{\tau} d\tau = e^{\frac{2}{\sqrt{S(T)}}} - e^{\frac{2}{\sqrt{\tilde{S}(t)}}}, \quad t \geq T \tag{3.54}$$

[setting $\tau = u^{-1}, d\tau = -u^{-2}du$]. We obtain from (3.54)

$$e^{\frac{2}{\sqrt{\tilde{S}(t)}}} = e^{\sqrt{\frac{4}{S(T)}}} + \frac{2(t-T)}{3},$$

or

$$\tilde{S}(t) = \frac{4}{\ln^2\left[e^{\sqrt{\frac{4}{S(T)}}} + \frac{2(t-T)}{3}\right]} \leq \frac{4}{\ln^2\left[e^{\sqrt{\frac{4}{E(0)}}} + \frac{2(t-T)}{3}\right]}, \ t \geq T, \tag{3.55}$$

since $S(0) = E(0) > S(T) = \tilde{S}(T)$, as $S(t) \searrow 0$ by [L-T.9, (B.3)] or (3.16).

By (3.36) of Corollary 3.5, we conclude that: with $g(z) = \gamma(|z|)z = z|z|e^{-\frac{1}{|z|}}$, for $|z|$ small, and otherwise subject to assumption (H.2), the solutions of the corresponding w-problem (3.3) satisfy the following energy decay rates

$$E(t) \leq C(E(0))S(t) \leq C(E(0)\tilde{S}(t) = C(E(0))\left\{\ln^2\left[e^{\sqrt{\frac{4}{E(0)}}} + \frac{2(t-T)}{3}\right]\right\}^{-1}, \tag{3.56}$$

$t \geq T$.

Example #3. In this illustration, we take the *linear* case near the origin:

$$\gamma(s) \equiv 1, \text{ or } g(s) = s \text{ near } s = 0, \text{ say } 0 \leq s \leq s_0; \ g(z) = z; \ x = g^{-1}(x). \tag{3.57}$$

In this case, all the relevant quantities in the algorithm: $h(\,\cdot\,)$, $p(\,\cdot\,)$, $q(\,\cdot\,)$ are directly computable.

Computation of $h(\,\cdot\,)$. By (3.11) we define $h(\,\cdot\,)$ by imposing for $|z|$ small, or s small

$$h(g(z)\bar{z}) = |z|^2 + |g(z)|^2 = 2|z|^2; \ h(g(s)s) = h(s^2) = 2s^2; \ h(y) = 2y, \ h^{-1}(x) = \frac{x}{2}, \tag{3.58}$$

near $x = 0$.

Computation of $p(x)$. By (3.13) with $\tilde{h} = h$ and $K = 1$, we have via (3.57):

$$Cp(x) + h(p(x)) = x, \text{ or } Cp(x) + 2p(x) = x, \text{ or } p(x) = \frac{x}{2+C} \text{ near } x = 0. \tag{3.59}$$

Computation of $q(x)$. By (3.14), we have by (3.59),

$$q(x) + p(q(x)) = p(x), \text{ or } q(x) + \frac{q(x)}{2+C} = \frac{x}{2+C}, \text{ or } q(x) = \frac{x}{3+C} \text{ near } x = 0. \tag{3.60}$$

ODE (3.17). With $q(\cdot)$ given by (3.60), the ODE (3.17) is

$$S_t(t) + \frac{1}{3+C} S(t) \equiv 0, \; S(0) = E(0), \text{ or } S(t) = E(0)e^{-\frac{t}{3+C}}, \; t > 0. \tag{3.61}$$

We apply Theorem 3.3, Eqn. (3.16), and we conclude that: with $g(z) = \gamma(|z|)z = z$, for $|z|$ small, and otherwise subject to assumption (H.2), the solutions of the corresponding w-problem (3.3) satisfy the following energy decay rates

$$E(t) \le C(E(0))S(t) = C(E(0))e^{-\frac{t}{3+C}}, \; t > 0. \tag{3.62}$$

Example #4. In this illustration we take near the origin:

$$\gamma(s) = \frac{1}{s^r} \text{ or } g(s) = s^{1-r}, \; 0 < r < 1, \; 0 < s \le s_0; \; g(z) = \frac{1}{|z|^r} z, \; |z| \text{ small}, \tag{3.63}$$

$g^{-1}(y) = y^{\frac{1}{1-r}}$.

Application of Corollary 3.6. By (3.63), [L-T.9, (3.28), (3.18)], we have

$$\gamma(0) = \infty; \; g'(0) = \infty; \; h'(0) = \infty; \; p'(0) = \frac{1}{C + h'(0)} = 0. \tag{3.64}$$

Thus, the definition of $\tilde{q}(x)$ in (3.41) is, by (3.63), near $x = 0$:

$$\tilde{q}(x) = \frac{2}{3} h^{-1}\left(\frac{x}{2}\right) = \frac{2}{3}\sqrt{\frac{x}{4}}\, g^{-1}\left(\sqrt{\frac{x}{4}}\right) = \frac{2}{3}\left(\frac{x}{4}\right)^m, \; 1 < m = \frac{1}{2}\left(1 + \frac{1}{1-r}\right) < \infty. \tag{3.65}$$

Then, we need to integrate Eqn. (3.42),

$$\tilde{S}_t(t) + \tilde{q}(\tilde{S}(t)) \equiv 0 \text{ or } \tilde{S}_t(t) + \frac{2}{3}\left(\frac{\tilde{S}(t)}{4}\right)^m = 0, \; \tilde{S}(T) = S(T), \tag{3.66}$$

for T sufficiently large. We obtain by separation ((3.44))

$$\frac{2}{3}(t-T) = -4\int_{S(T)}^{\tilde{S}(t)} \left(\frac{\tilde{S}}{4}\right)^{-m} d\left(\frac{\tilde{S}}{4}\right) = \frac{4}{m-1}\left[\left(\frac{\tilde{S}(t)}{4}\right)^{1-m} - \left(\frac{S(T)}{4}\right)^{1-m}\right]; \tag{3.67}$$

$$\tilde{S}(t) = \frac{4}{\left[\left(\frac{4}{S(T)}\right)^{m-1} + \frac{m-1}{6}(t-T)\right]^{\frac{1}{m-1}}}, \quad \forall\, t \ge T \tag{3.68}$$

$$\le \frac{4}{\left[\left(\frac{4}{E(0)}\right)^{m-1} + \frac{m-1}{6}(t-T)\right]^{\frac{1}{m-1}}}, \quad \forall\, t \ge T, \tag{3.69}$$

since $S(0) = E(0) > S(T) = \tilde{S}(T)$ as $S(t) \searrow 0$ as $t \to \infty$ by (3.16) or [L-T.9, (B.3)]. By (3.43) of Corollary 3.6 we conclude that: with $g(z) = \frac{1}{|z|^r} z$, $0 < r < 1$, $|z|$ small, and otherwise subject to assumption (H.2), the solutions of the corresponding w-problem (3.3) satisfy the following energy decay rates

$$\begin{aligned} E(t) &\leq C(E(0))S(t) \leq C(E(0))\tilde{S}(t) \\ &\leq C(E(0)) \left[\left(\frac{4}{E(0)} \right)^{m-1} + \frac{m-1}{4}(t-T) \right]^{1-m}, \ \forall\, t \geq T. \quad \square \end{aligned} \tag{3.70}$$

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R. Triggiani
Department of Mathematics
University of Virginia
Charlottesville, VA 22903- USA
E-mail address: rt7u@virginia.edu